ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2015, Vol. 55, No. 9, pp. 1547–1553. © Pleiades Publishing, Ltd., 2015.
Original Russian Text © N.Ya. Moiseev, E.A. Shestakov, 2015, published in Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 2015, Vol. 55, No. 9, pp. 1579–1585.

Solution of the Riemann Problem in Two- and Three-Temperature Gas Dynamics

N. Ya. Moiseev and E. A. Shestakov

All-Russia Research Institute of Technical Physics, Russian Federal Nuclear Center, ul. Vasil'eva 13, Snezhinsk, Chelyabinsk oblast, 456770 Russia

e-mail: nyamoiseyev@vniitf.ru

Received September 3, 2014; in final form, February 19, 2015

Abstract—Approximate and exact Riemann solvers in two- and three-temperature gas dynamics are presented. The numerical methods are designed without constructing a general equation of state for the medium and are based on the solution of the Riemann problem in one-temperature gas dynamics.

DOI: 10.1134/S0965542515090110

Keywords: Godunov's method, Riemann problem in two- and three-temperature gas dynamics.

1. INTRODUCTION

Godunov's method [1] is widely used for the numerical solution of continuum mechanics problems. An important role in the method is played by the solution of the Riemann problem arising at the interface of two neighboring cells. These solutions can be approximate or exact (see [2, 3]). Wide experience has been accumulated to date concerning the numerical solution of problems by applying this method, in particular, in gas dynamics.

In [4–6] Godunov's method is generalized to simulate processes in microscopic targets under extremal compression and heating. The medium (plasma) is described as a gas with a uniform density and particle velocity. This model is known as a *single-liquid* one. It can be one-, two-, or three-temperature; in the last case, the particles of all types have different temperatures. Numerical solutions are found by applying the splitting method over physical processes. An example is a gasdynamic process described by the system of differential equations of two- or three-temperature gas dynamics. Solutions of the equations are found using a difference scheme based on Godunov's method. To solve the Riemann problem, a generalized two-term equation of state for the medium is constructed. Next, the solution is found by well-known methods in the same manner as in usual gas dynamics (see [3, 7]). This approach preserves the conceptual and algorithmic basis for the construction of difference schemes and makes it possible to use methodological constructions and software tools available in gas dynamic applications. However, Riemann solvers based on such an equation of state are rather complicated and expensive [5].

In this paper, Riemann solvers for two- or three-temperature gas dynamics are designed on the basis of solving the Riemann problem in one-temperature gas dynamics without constructing an equation of state for the medium.

2. FORMULATION OF THE PROBLEM

Consider a medium (plasma) consisting of particles of three types: ions, electrons, and photons, which are responsible for radiative energy transfer and are denoted by the indices i, e, and f, respectively. The medium is treated as a gas (or fluid) with a uniform particle density ρ and a uniform velocity $\mathbf{u}$. The temperatures T_k, pressures p_k, and specific internal energies ε_k of the particles are assumed to depend on space (x) and time t. Here and below, $k = i, e, f$.

Following [4], the system of differential equations of motion at the gasdynamic stage in the case of plane symmetry in the three-temperature approximation is written as

$$\begin{aligned} &\frac{d\rho}{dt} + \rho\frac{\partial u}{\partial x} = 0,\\ &\frac{du}{dt} + \frac{1}{\rho}\frac{\partial \hat{p}}{\partial x} = 0,\\ &\frac{d\varepsilon_k}{dt} + p_k\left(\frac{1}{\rho}\frac{\partial u}{\partial x}\right) = 0. \end{aligned} \tag{2.1}$$

Here, $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ is the substantial derivative; t and x are independent time and space variables, respectively; ρ is the specific gas density; u is the velocity of the gas; p_k and ε_k are the pressures and specific internal energies of the ions, electrons, and photons, respectively; and $\hat{p} = p_i + p_e + p_f$ is the total pressure. Quantities without indices are gas parameters. The system of equations (2.1) is closed by the equations of state $p_k = p_k(\rho, \varepsilon_k)$ for each component. In view of the equations of state, the energy equations are written as (see [8])

$$\frac{dp_k}{dt} + a_k^2\frac{1}{\rho}\frac{\partial u}{\partial x} = 0, \quad a_k^2 = \frac{p_k + (\varepsilon_k)_\nu}{(\varepsilon_k)_p}, \tag{2.2}$$

where ε_ν and ε_p are partial derivatives with respect to ν and p and $\nu = 1/\rho$ is the specific volume of the gas. Summing Eqs. (2.2) yields the equation

$$\frac{d\hat{p}}{dt} + \hat{a}^2\frac{1}{\rho}\frac{\partial u}{\partial x} = 0$$

for the gas pressure, where $\hat{a}^2 = a_i^2 + a_e^2 + a_f^2$. The pressure and internal energy of radiation (photons) are related by the formula

$$\rho\varepsilon_f = 3p_f. \tag{2.3}$$

From (2.3), we can derive an equation of state for radiation in the ideal gas form

$$p_f = (\gamma_f - 1)\rho\varepsilon_f, \quad \gamma_f = 4/3. \tag{2.4}$$

Therefore, radiation can be treated as a gas with the ratio of specific heats $\gamma_f = 4/3$.

In the two-temperature approximation, when the plasma is a gas with radiation, three energy equations in (2.1) are replaced by two equations

$$\begin{aligned} &\frac{d\varepsilon}{dt} + p\left(\frac{1}{\rho}\frac{\partial u}{\partial x}\right) = 0,\\ &\frac{d\varepsilon_f}{dt} + p_f\left(\frac{1}{\rho}\frac{\partial u}{\partial x}\right) = 0, \end{aligned}$$

where p and ε are the pressure and specific internal energy of the gas, respectively. If the plasma is a gas of ions and electrons [9], we use the equations

$$\begin{aligned} &\frac{d\varepsilon_i}{dt} + p_i\left(\frac{1}{\rho}\frac{\partial u}{\partial x}\right) = 0,\\ &\frac{d\varepsilon_e}{dt} + p_e\left(\frac{1}{\rho}\frac{\partial u}{\partial x}\right) = 0. \end{aligned}$$

At $t = 0$, the medium parameters in the Riemann problem are indexed by 1 and 2 for $x < 0$ and $x > 0$, respectively. The components of the media obey their equations of state, which are considered in the form of two-term equations:

$$\varepsilon_k = \frac{p_k + \gamma_k p_k^0}{(\gamma_k - 1)\rho} - \frac{c_{0k}^2}{\gamma_k - 1}, \quad \left(p_k = (\gamma_k - 1)\rho\varepsilon_k + c_{0k}^2\rho - \gamma_k p_k^0\right), \tag{2.5}$$

where p_k^0, γ_k, and c_{0k}^2 are the medium parameters. According to (2.4), for radiation, we have $p_f^0 = c_{0f}^2 = 0$ and $\gamma_f = 4/3$. The media are divided by a barrier that instantaneously disappears at the time $t = 0$. The task is to determine the flow at $t > 0$.

If the equation of state is other than a two-term one, we approximate it by such an equation. The parameters of the approximating equation are given by the formulas

$$\gamma = 1 + \frac{1}{\rho}\frac{\partial p}{\partial \varepsilon}, \quad p_0 = \frac{1}{\gamma}\left(\rho\frac{\partial p}{\partial \rho} - p\right), \quad c_0^2 = \frac{\partial p}{\partial \rho} - \frac{\varepsilon}{\rho}\frac{\partial p}{\partial \varepsilon},$$

which were proposed by A.V. Zabrodin. Here, p, ρ, and ε are related by the given equation of state.

Since our study relies on [4, 5], we preliminarily present the basic equations from [4, 5] to be used to solve the Riemann problem. The self-similar configuration arising at $t > 0$ is the same as in usual gas dynamics. An analysis of the characteristic equation for system (2.1) shows that its roots

$$\lambda_1 = u - \hat{c}, \quad \lambda_2 = u + \hat{c}, \quad \lambda_3 = u$$

are real. The root $\lambda_3 = u$ is of multiplicity 3 and corresponds to a particle trajectory. Here,

$$\hat{c}^2 = c_i^2 + c_e^2 + c_f^2, \quad c_k = \frac{\partial p_k}{\partial \rho} + \frac{p_k}{\rho^2}\frac{\partial p_k}{\partial \varepsilon_k}.$$

For two-term equations of state, $c_k^2 = \gamma_k\left(p_k + p_k^0\right)/\rho$. By using the relations on the three-fold characteristic-trajectory, for each component of the medium, we can introduce its entropy function

$$\sigma_k = \left(p_k + p_k^0\right)/\rho^{\gamma_k}.$$

For smooth solutions, the entropy of each component remains constant ($\sigma_k = \text{const}$) along the trajectory $(dx/dt) = u$. Along the other two (sonic) characteristics $(dx/dt) = u \pm \hat{c}$, we have the relations

$$du \pm \frac{d\hat{p}}{\rho\hat{c}} = 0$$

respectively. Summing the energy equations in (2.1) over all k and adding the result to the second equation multiplied by u, for the total energy of the gas, we obtain the equation

$$\frac{d\left(\hat{\varepsilon} + 0.5u^2\right)}{dt} + \frac{1}{\rho}\frac{\partial(\hat{p}u)}{\partial x} = 0, \quad \hat{\varepsilon} = \varepsilon_i + \varepsilon_e + \varepsilon_f,$$

which is used to compute shock waves (SWs). Following [5], the formulas for SWs are written as

$$\begin{gathered} U - u_{1,2} \pm \frac{\hat{P} - \hat{p}_{1,2}}{\hat{a}_{1,2}} = 0, \quad \hat{a}_{1,2} = \sqrt{\left(\hat{P} - \hat{p}_{1,2}\right)/\left(\frac{1}{\rho_{1,2}} - \frac{1}{R_{1,2}}\right)}, \quad D_{1,2} = u_{1,2} \mp \frac{\hat{a}_{1,2}}{\rho_{1,2}}, \\ \hat{E}_{1,2} - \hat{\varepsilon}_{1,2} - \left(\frac{1}{\rho_{1,2}} - \frac{1}{R_{1,2}}\right)\left(\frac{\hat{P} + \hat{p}_{1,2}}{2}\right) = 0, \quad E_{k1,2} - \varepsilon_{k1,2} - \left(\frac{1}{\rho_{1,2}} - \frac{1}{R_{1,2}}\right)\left(\frac{P_k + p_k}{2}\right)_{1,2} = 0. \end{gathered} \tag{2.6}$$

Here, as in Godunov's scheme, the capital and lowercase letters denote the quantities behind and ahead of the wavefront, respectively. The index 1 (2) and the uppercase (lowercase) letters correspond to the left (right) SW. The first four formulas are derived from the Rankine–Hugoniot conditions. Since the variation in the total energy of the gas (fourth equation) has an additive form, the variation in the energy of each component is naturally written in the same form (this is the fifth equation). In terms of the uniform density R behind the SW front, the shock Hugoniot curves for the components with equations of state of form (2.5) are written as

$$P_k = \frac{\left[(\gamma_k + 1)R - (\gamma_k - 1)\rho\right]p_k + 2\gamma_k(R - \rho)p_k^0}{(\gamma_k + 1)\rho - (\gamma_k - 1)R}, \quad \frac{R}{\rho} < \frac{\gamma_k + 1}{\gamma_k - 1}. \tag{2.7}$$

If the left wave is a rarefaction wave (RW), then expressions for computing the pressures are derived from the condition that the entropy functions of the components are preserved along the trajectory:

$$P_k = \left(p_k + p_k^0\right)\left(\frac{R}{\rho}\right)^{\gamma_k} - p_k^0. \tag{2.8}$$

The exact solution of the Riemann problem for complicated equations of state can be obtained using an iterative method, for example, one of the methods described in [2, 10]. In what follows, we assume that $\hat{p}_1 \le \hat{p}_2$. If this condition is not satisfied, then the indices of the original parameters and the velocity signs are interchanged. After the problem is solved, an inverse operation is performed.

3. APPROACH TO THE EXACT SOLUTION OF THE RIEMANN PROBLEM IN TWO- AND THREE-TEMPERATURE MEDIA

The solution algorithm for the Riemann problem in two- and three-temperature gas dynamics is similar to that in the case of one-temperature gas dynamics. However, there is a feature in the case of an RW-involving configuration. Accordingly, we first describe this feature.

Suppose that a configuration with two RWs arises in the Riemann problem. Then the problem is solved using the constants on the corresponding characteristics

$$\frac{dx}{dt} = u \pm \hat{c}(\rho)$$

of the Riemann invariant (see [11])

$$u \pm \sigma(\rho) = \text{const}, \quad \sigma(\rho) = \int_{\rho_{1,2}}^{\rho} \frac{\hat{c}(\rho)}{\rho} d\rho,$$

where the plus (minus) sign corresponds to the index 1 (2). Since the velocity and pressure on the contact discontinuity (CD) are continuous, in view of (2.8), we obtain a system of two equations with two unknowns R_1 and R_2:

$$\int_{R_1}^{\rho_1} \frac{\hat{c}(\rho)}{\rho} d\rho + \int_{R_2}^{\rho_2} \frac{\hat{c}(\rho)}{\rho} d\rho = u_1 - u_2,$$

$$\left(P_i + P_e + P_f\right)_1 = \left(P_i + P_e + P_f\right)_2.$$

In usual gas dynamics for a gas with a two-term equation of state, the integral can be evaluated analytically (see [7, 9]), but this cannot be done in two- and three-temperature gas dynamics. Nevertheless, the integral can be approximately evaluated via integrals in one-temperature gas dynamics if we perform the following transformations. Multiplying and dividing the integrand by $\hat{c}$ gives

$$\int \frac{\hat{c}}{\rho} d\rho = \sum_{k=i,e,f} \int \alpha_k \frac{c_k}{\rho} d\rho, \quad \alpha_k = \frac{c_k}{\hat{c}} = \sqrt{\frac{\gamma_k\left(p_k + p_k^0\right)}{\sum \gamma_k\left(p_k + p_k^0\right)}}.$$

Assume that the coefficients α_k vary weakly along the corresponding Poisson adiabat. If this is not the case, then the interval of integration with respect to density is partitioned so that the coefficients α_k vary weakly on each interval. Then the integral can be represented as a sum of integrals in one-temperature gas dynamics and can be integrated by quadratures as

$$\sigma(\hat{c}) = \sum_k \alpha_k \int \frac{c_k}{\rho} d\rho = \sum_k \alpha_k \frac{2}{\gamma_k - 1} c_k.$$

In view of (2.8), the CD velocity is given by the formulas

$$U = u_{1,2} \pm \sum_{k=i,e,f} \alpha_k \frac{2}{\gamma_k - 1}(c_{k1,2} - C_{k1,2}) = u_{1,2} + \left(\sum_{k=i,e,f} \alpha_k \frac{2}{\gamma_k - 1} c_k \left[1 - z^{\frac{\gamma_k - 1}{2}}\right]\right)_{1,2}, \quad z_{1,2} = \frac{R_{1,2}}{\rho_{1,2}}$$

with the upper (lower) sign and index 1 (2) used for the left (right) RW. The expressions for the differences $(c_{k1,2} - C_{k1,2})$ are taken from [3, p. 108].

By following this approach to integral evaluation, we write the following system of two equations with two unknowns $z_1 = R_1/\rho_1$ and $z_2 = R_2/\rho_2$ for computing the velocities and pressures on the CD (this system is the same for all configurations):

$$f_1(z_1) + f_2(z_2) = u_1 - u_2,$$
$$g_1(z_1) - g_2(z_2) = 0,$$

where the functions are defined according to the arising configuration. Specifically, the functions are given by

$$f_{1,2} = \left(\sqrt{\sum_k \frac{2\gamma_k \left(p_k + p_k^0\right)(z-1)^2}{\rho\left[(\gamma_k+1)z - (\gamma_k - 1)z^2\right]}} \right)_{1,2},$$

$$g_{1,2} = \left(\sum_{k=i,e,f} \frac{\left[(\gamma_k+1)z - (\gamma_k-1)\right]p_k + 2\gamma_k (z-1)\, p_k^0}{(\gamma_k+1) - (\gamma_k - 1)z} \right)_{1,2}$$

for a configuration of two SWs,

$$f_{1,2} = \left(\sum_{k=i,e,f} \alpha_k \frac{2}{\gamma_k - 1} c_k \left[1 - z^{\frac{\gamma_k - 1}{2}} \right] \right)_{1,2}, \quad g_{1,2} = \left(\sum_{k=i,e,f} \left(p_k + p_k^0\right) z^{\gamma_k} - p_k^0 \right)_{1,2}$$

for a configuration of two RWs, and

$$f_1 = \left(\sqrt{\frac{1}{\rho} \sum_{k=i,e,f} \frac{2\gamma_k \left(p_k + p_k^0\right)(z-1)^2}{(\gamma_k+1)z - (\gamma_k-1)z^2}} \right)_1, \quad f_2 = \left(\sum_{k=i,e,f} \alpha_k \frac{2}{\gamma_k - 1} c_k \left[1 - z^{\frac{\gamma_k - 1}{2}} \right] \right)_2$$

for a configuration of an SW and an RW. Solving the system of equations by Newton's method, we find the densities R_1 and R_2. In a SW, the other quantities are calculated using formulas (2.6) and (2.7). In an RW, the pressures are given by (2.8), while the speed of sound $C_{k1,2}$ is found using the formula

$$C_{k1,2}^2 = \gamma_{k1,2}\left(P_{k1,2} + p_{k1,2}^0\right)/R_{1,2}.$$

The velocities of the extreme RW characteristics are given by the formulas

$$(D_1)_{1,2} = u_{1,2} \mp \hat{c}_{1,2}, \quad (D_2)_{1,2} = U \mp \hat{C}_{1,2}.$$

The algorithm for determining the configuration occurring in the Riemann problem is similar to that used in [3, p. 112] with the pressures p and P replaced by the total pressures $\hat{p}$ and $\hat{P}$, respectively. Then the pressure function at reference points is given by the formulas

$$F(\hat{p}_2) = U_{\mathrm{SW}} = \frac{\hat{p}_2 - \hat{p}_1}{\hat{a}_1(\hat{p}_2)},$$

$$F(\hat{p}_1) = U_{\mathrm{RW}} = -\sum_{k=i,e,f} \left(\alpha_k \frac{2}{\gamma_k - 1} c_k \right)_2 \left[1 - \left(\frac{p_k + p_k^0}{p_k + p_k^0} \right)^{\frac{\gamma_k - 1}{2\gamma_k}} \right]_2,$$

$$F\left(-p_k^0\right) = U_{\mathrm{VAC}} = -\sum_{k=i,e,f} \left(\alpha_k \frac{2}{\gamma_k - 1} c_k \right)_1 - \sum_{k=i,e,f} \left(\alpha_k \frac{2}{\gamma_k - 1} c_k \right)_2.$$

The configuration developing in the Riemann problem is determined by analyzing the following conditions:

1. If $u_1 - u_2 > U_{SW}$, then $\hat{P} > \hat{p}_2$ and $\hat{P} > \hat{p}$.
2. If $U_{RW} < u_1 - u_2 < U_{SW}$, then $\hat{p}_1 < \hat{P} < \hat{p}_2$.
3. If $U_{VAC} < u_1 - u_2 < U_{RW}$, then $\hat{p}_0 < \hat{P} < \hat{p}_1$.
4. If $u_1 - u_2 < U_{VAC}$, then $\hat{P} = -\hat{p}_0$.

The configuration consists of two SWs in case 1, of an SW and an RW in case 2, of two RWs in case 3, and a vacuum region forms in case 4.

4. APPROACH TO THE APPROXIMATE SOLUTION OF THE RIEMANN PROBLEM IN TWO- AND THREE-TEMPERATURE MEDIA

It is well known that approximate solutions of the Riemann problem are used to solve gasdynamic problems by applying Godunov's method in regions with weakly varying flow parameters. Naturally, it is desirable to apply similar approximate solutions in corresponding situations in two- and three-temperature media. By solving linearized equations (2.1) and (2.2), an approximate "sonic" solution of this kind can be obtained in the form

$$\hat{P} = \frac{\hat{a}_2\hat{p}_1 + \hat{a}_1\hat{p}_2}{\hat{a}_2 + \hat{a}_1} - \hat{a}_2\hat{a}_1\frac{u_2 - u_1}{\hat{a}_2 + \hat{a}_1},$$

$$U = \frac{\hat{a}_2u_2 + u_1\hat{a}_1}{\hat{a}_2 + \hat{a}_1} - \frac{\hat{p}_2 - \hat{p}_1}{\hat{a}_2 + \hat{a}_1}, \quad \hat{a}_{1,2}^2 = \left(a_i^2 + a_e^2 + a_f^2\right)_{1,2}.$$

Approximate solutions are used as initial data for finding solutions by Newton's iterative method. In the RW computations, the initial approximation is calculated using the formulas

$$P^0 = 0.5\left(\hat{P} + \frac{\hat{p}_1 + \hat{p}_2}{2}\right), \quad U^0 = 0.5\left(U + \frac{u_1 + u_2}{2}\right).$$

5. NUMERICAL RESULTS FOR TEST PROBLEMS

Below, we present the solutions obtained for three model Riemann problems in three-temperature gas dynamics for three possible configurations. The coefficients α_k in RW were calculated from the initial data. In all the examples, the ion and electron components are ideal gases with $\gamma_i = 3$ and $\gamma_e = 1.4$.

The collision of two flows with initial parameters $\rho_{1,2} = 1.0$, $u_{1,2} = \pm 1.0$, $p_{i1,2} = 0.9$, $p_{e1,2} = 0.35$, $p_{f1,2} = 0.75$, and $\hat{c} = 2.047$ was calculated in problem I. Problem II was concerned with homogeneous flows in the plane $x = 0$ with initial parameters $\rho_{1,2} = 1.0$, $u_{1,2} = \mp 1.0$, $p_{i1,2} = 0.85$, $p_{e1,2} = 0.5$, $p_{f1,2} = 0.65$, and $\hat{c} = 2.0290$. In problem III, the initial parameters of fixed gases were specified as $\rho_1 = 1.0$, $\rho_2 = 0.125$, $u_{1,2} = 0$, $p_{i1} = 0.45$, $p_{i2} = 0.045$, $p_{e1} = 0.35$, $p_{e2} = 0.035$, $p_{f1} = 0.2$, $p_{f2} = 0.02$, $\hat{c}_1 = 1.2982$, and $\hat{c}_2 = 1.4514$. The resulting solutions are presented in the table.

Results of solving the model problems

Problem, side of the CD	R	U	P_i	P_e	P_f	$\hat{C}$	$D1$	$D2$
I, 1, 2	1.4681	0	3.2758	0.6030	1.2577	2.9002	–	–
II, 1, 2	0.5512	0	0.1423	0.2172	0.2938	1.4272	$\mp$3.0289	$\mp$1.4272
III, 1	0.5326	0.7445	0.0680	0.1449	0.0863	0.9899	–1.4514	–0.2454
III, 2	0.1917	0.7445	0.1993	0.0643	0.0356	1.9587	–	–

6. CONCLUSIONS

Exact (in the case of two-term equations of state) and approximate Riemann solvers in two- and three-temperature gas dynamics were presented. A feature of the algorithms is that solutions are found without constructing a general equation of state for the medium. The resulting solutions can be useful in the numerical solution of problems in two- and three-temperature gas dynamics.

REFERENCES

1. S. K. Godunov, "Finite-difference method for calculating discontinuous solutions to fluid dynamics equations," Mat. Sb. **47** (3), 271–306 (1959).
2. G. B. Alalykin, S. K. Godunov, I. L. Kireeva, et al., *Solution of One-Dimensional Problems in Gas Dynamics on Moving Meshes* (Nauka, Moscow, 1970) [in Russian].
3. *Numerical Solution of Multidimensional Problems in Gas Dynamics*, Ed. by S. K. Godunov (Nauka, Moscow, 1976) [in Russian].
4. A. V. Zabrodin and G. P. Prokopov, "Method for the numerical modeling of plane unsteady flows of heat-conducting gas in three-temperature approximation," Vopr. Atom. Nauki Tekh., Ser. Mat. Model. Fiz. Protsessov **3**, 3–16 (1998).
5. G. P. Prokopov, Preprint No. 66, IPM RAN (Keldysh Inst. of Applied Mathematics, Russian Academy of Sciences, Moscow, 2004).
6. G. P. Prokopov, Preprint No. 32, IPM RAN (Keldysh Inst. of Applied Mathematics, Russian Academy of Sciences, Moscow, 2000).
7. G. P. Prokopov, "Computation of a discontinuity for porous media and continuum media with a two-term equation of state," Vopr. Atom. Nauki Tekh., Ser. Metod. Program. Chisl. Reshen. Zadach Mat. Fiz. **2** (10), 32–40 (1982).
8. B. L. Rozhdestvenskii and N. N. Yanenko, *Systems of Quasilinear Equations and Their Applications to Gas Dynamics* (Nauka, Moscow, 1968; Am. Math. Soc., Providence, 1983).
9. A. G. Kulikovskii, N. V. Pogorelov, and A. Yu. Semenov, *Mathematical Aspects of Numerical Solution of Hyperbolic Systems* (Fizmatlit, Moscow, 2001; Chapman and Hall/CRC, London, 2001).
10. N. Ya. Moiseev and T. A. Mukhamadieva, "Newton's method as applied to the Riemann problem for media with general equations of state," Comput. Math. Math. Phys. **48** (6), 1039–1047 (2008).
11. L. V. Ovsyannikov, *Lectures on the Foundations of Gas Dynamics* (Nauka, Moscow, 1981) [in Russian].

Translated by I. Ruzanova